Probability Distributions and Estimators for Multipath Fading Channels

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The purpose of this paper is to provide a concise reference for the distributions and estimators of the mean for Rayleigh and exponential random variables. These random variables play an important role in land mobile radio because they accurately describe the instantaneous amplitude and power, respectively, of a multipath fading signal.

1.0 Rayleigh Distribution

Using central limit theorem arguments, one can show that the *I* and *Q* channels on a mobile radio multipath fading channel are independent Gaussian (normal) random variables. Jakes [1] and others show that the envelope of two independent and identically distributed (iid) Gaussian random variables is Rayleigh distributed.[1]

Probability Density Function (pdf) (usual form for mobile radio applications):

$$f_X(x) = \frac{2x}{s^2} e^{-x^2/s^2}, \;\; x \geq 0 \tag{1}$$

where $s^2/2 = \sigma^2$ is the variance of the each of the original Gaussian random variables.

Cumulative Distribution Function (cdf):

$$F_X(x) = 1 - e^{-x^2/s^2}, \;\; x \geq 0 \tag{2}$$

Note from (2) that if the amplitude is Rayleigh-distributed, the power, which is the square of the amplitude, is exponentially distributed with mean s^2.

Mean:

$$\mu = \frac{\sqrt{\pi}}{2} \; s \tag{3}$$

Standard Deviation:

$$\sigma = \sqrt{1 - \frac{\pi}{4}} \; s \tag{4}$$

[1]By envelope, we mean the square root of the sum of the squares.

Root Mean Square (rms) Value:

$$Root \;\; Mean \;\; Square = \sqrt{E[X^2]} = s \tag{5}$$

Median:

$$Median = \sqrt{\frac{\ln(4)}{2}} \;\; s \tag{6}$$

2.0 Exponential Distribution

Probability Density Function:

$$f_X(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), \;\; x \geq 0 \tag{7}$$

Cumulative Distribution Function:

$$F_X(x) = 1 - \exp\left(-\frac{x}{\theta}\right), \;\; x \geq 0 \tag{8}$$

Mean:

$$\mu = \theta \tag{9}$$

Standard Deviation:

$$\sigma = \theta \tag{10}$$

Root Mean Square (rms) Value:

$$Root \;\; Mean \;\; Square = \sqrt{2}\theta \tag{11}$$

Median:

$$Median = -\ln\left(\frac{1}{2}\right)\theta \tag{12}$$

Note: The median of the exponential is 1.6 dB below the mean. The median of the Rayleigh is 0.54 dB below the mean. The ratio of the exponential mean to the Rayleigh mean is 1.05 dB.

Table 1 lists the probability that an independent sample of the exponential distribution will fall below a particular level with respect to the mean and with respect to the median. Note that a simple rule of thumb applies to the mean: roughly 10% of the samples are more than 10 dB below the mean,1% are more than 20 dB below the mean, and 0.1% are more than 30 dB below the mean.

Table 1 - CDF for Exponential Distribution

Level w.r.t. Mean	Probability	Level w.r.t. Median	Probability
+6 dB	0.98	+6 dB	0.94
+3 dB	0.86	+3 dB	0.75
0 dB	0.63	0 dB	0.50
-10 dB	9.52E-02	-10 dB	6.70E-02
-20 dB	9.95E-03	-20 dB	6.91E-03
-30 dB	1.00E-03	-30 dB	6.93E-04
-40 dB	1.00E-04	-40 dB	6.93E-05

3.0 Estimators for the Mean

Mobile radio receivers, either analog or digital, are specified to achieve a service threshold at a particular *mean* signal level in multipath fading, assuming the channel is thermal noise limited. The service threshold is typically 12 dB SINAD for analog FM receivers and a bit-error rate of 10^{-3} for digital receivers.[2] Note that the required carrier-to-noise ratio (*C/N*) to achieve this service threshold is significantly higher on a fading channel than a static channel. For example, an FM receiver with a peak deviation of 5 kHz (25 kHz channel) requires a *C/N* of 20 dB in Rayleigh fading but only 4 dB on a static channel [8].

We must first decide whether we are estimating the mean amplitude of the channel or the mean power. We will start with amplitude. In [2], Lee derives an expression for the mean amplitude of the Rayleigh fading channel assuming independent samples and using the central limit theorem to approximate the estimator as a normal random variable. The central limit theorem can be stated as follows:

Central limit Theorem: Let X_1, X_2, ... be a sequence of independent and identically distributed random variables, each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + ... + X_n - n\mu}{\sigma\sqrt{n}} \qquad (13)$$

tends to the standard normal as n approaches infinity. That is,

[2]A value of 5% errors is also used for digital voice receivers.

$$P\left(\frac{X_1 + ... + X_n - n\mu}{\sigma\sqrt{n}} \le a\right) \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx \qquad \text{as } n \rightarrow \infty \tag{14}$$

The central limit theorem leads one to consider the following estimator:

$$W = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{15}$$

This estimator makes it convenient to compute the confidence interval of the estimate using the central limit theorem. Specifically, we can re-write (14) in terms of the estimator, *W*, as

$$P\left(\frac{W - \mu}{\sigma/\sqrt{n}} \le a\right) \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx \qquad \text{as } n \rightarrow \infty \tag{16}$$

Unbiased Estimators. An estimator is not automatically a good estimator simply because it allows us to conveniently use the central limit theorem. Typically, good estimators are *unbiased*, meaning that the mean of the estimator equals the mean of the random variable being estimated.[3]

Peritsky [6] and others show that an unbiased estimator with low variance for the Rayleigh mean involves the square root of the sum the squares of the samples, not the arithmetic mean of the samples.

But should we even bother with the Rayleigh distribution? The test receiver is probably not reporting the amplitude directly, so the Rayleigh distribution may be a poor assumption. In many cases, the test receiver delivers a logarithmic value of the received *power* (e.g., dBm). Typically, the user will convert each of these dBm readings to milliwatts, compute the arithmetic mean, and then convert back to dBm.[4] Thus, the user typically deals with power, which has an exponential distribution. In the next section, we will introduce an unbiased estimator for the exponential distribution and derive and exact expression for the confidence interval.

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[3]Estimators that are minimum variance, i.e., they achieve the Cramer-Rao lower bound, are also desirable, but for Rayleigh and exponential distributions, minimum variance estimators are more complex than the sample mean and beyond the scope of this paper. See [6] and [10] for more discussion on this subject.

[4]It is necessary to compute the mean from the linear values because of the well-known -2.5 dB bias error when using the logarithmic values directly [7, pp. 100].) Wong and Cox address a slightly different problem in [10] where they intentionally want to estimate not the mean signal power, but the mean of the logarithmic values of the signal power. In this case, the sample mean of the logarithmic values is unbiased (but not minimum variance).

4.0 Confidence Interval for Exponential Distribution

For a confidence level (CL) of (1-α), we want the values of c and d such that the following is true:

$$P(c < \theta < d) = 100(1-\alpha)\% \tag{17}$$

where θ is the mean power level were are trying to estimate. We know the probability density function (pdf) of the exponential random variable is the following:

$$f_X(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), \ \ x \geq 0 \tag{18}$$

Our proposed estimator is the sample average,

$$W = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{19}$$

We can prove this estimator is unbiased by recalling that the moment generating function of a random variable, $M_X(t)$, is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{20}$$

The expected value of X is found from the first derivative of $M_X(t)$ with respect to t, evaluated at t=0 [5, pp. 287]. I.e., $M'(0) = E[X]$. We also know that the sum of n iid exponential random variables, each with mean θ, is a gamma random variable with parameters (n,θ), The corresponding moment generating function is

$$M_Z(t) = \left(\frac{1/\theta}{1/\theta - t}\right)^n \tag{21}$$

and one can show that $M_Z'(t)$ is

$$M_Z'(t) = \frac{n}{\theta}\left(\frac{1}{1-\theta t}\right)^{n-1}\left(\frac{1}{\theta} - t\right)^{-2} \tag{22}$$

Evaluating this expression at t=0, we get

$$M_Z'(0) = \frac{n\theta^2}{\theta} = n\theta \tag{23}$$

and the mean of the estimator is

$$E(W) = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n}M_Z{}'(0) = \frac{n\theta}{n} = \theta \tag{24}$$

Thus, the estimator is unbiased. Now let's find the confidence interval for θ using this estimator. From [5, pp.289], we know that the moment generating function of an exponential random variable with mean θ is

$$M(t) = \frac{1/\theta}{1/\theta - t} \tag{25}$$

For reasons that will become clear in a moment, let's first find the distribution of the random variable, *Y*, defined as

$$Y = 2n\overline{X}/\theta \tag{26}$$

The moment generating function of *Y* is given by

$$M_Y(t) = M_{2n\overline{X}/\theta}(t) = M_{\overline{X}}[(2n/\theta)t] \tag{27}$$

We note that the moment generating function of the sum of *n* iid random variables is the n^{th} product of the moment generating function of the random variable. Thus,

$$M_{\overline{X}}[(2n/\theta)t] = M_{X_i}[2t/\theta]^n \tag{28}$$

Because *X* is exponential, we get

$$M_Y(t) = \left[\frac{1}{1-\theta(2t/\theta)}\right]^n = \left(\frac{1}{1-2t}\right)^n = \left(\frac{1/2}{1/2 - t}\right)^n \tag{29}$$

We see that this expression is the moment generating function for a gamma random variable with parameters (*n*, 1/2), also known as the Chi-squared distribution with 2*n* degrees of freedom [4]. The pdf for this random variable is

$$f_X(x) = \frac{\left(\frac{x}{2}\right)^{n-1}\frac{1}{2}e^{-x/2}}{\Gamma(n)},\ \ x \ge 0 \tag{30}$$

Now, let's return to our expression for the confidence interval:

$$CL = 100(1-\alpha)\% = P(c < \theta < d) = P\left(c < \frac{2n\overline{X}}{Y} < d \right) \tag{31}$$

$$= P\left(d < \frac{Y}{2n\overline{X}} < c \right) = P\left(\chi^2_{\alpha/2,2n} / 2n\overline{X} < \frac{Y}{2n\overline{X}} < \chi^2_{1-\alpha/2,2n} / 2n\overline{X} \right)$$

$$= P\left(2n\overline{X} / \chi^2_{1-\alpha/2,2n} < \theta < 2n\overline{X} / \chi^2_{\alpha/2,2n} \right) \tag{32}$$

where $\chi^2_{\alpha/2,2n}$ is the argument of the Chi-squared cdf with $2n$ degrees of freedom when the cdf value is $\alpha/2$ and $\chi^2_{1-\alpha/2,2n}$ is the argument when the cdf value is $1-\alpha/2$. The confidence interval for θ is just

$$CI(\theta) = \left(2n\overline{X} / \chi^2_{1-\alpha/2,2n}, 2n\overline{X} / \chi^2_{\alpha/2,2n} \right) \tag{33}$$

It is common to express the confidence interval as a ratio of $\overline{X}$ to θ or this ratio in decibels. The confidence interval for $\overline{X}/\theta$ is found from a straightforward algebraic manipulation:

$$CI\left(\frac{\overline{X}}{\theta} \right) = \left(\frac{\chi^2_{\alpha/2,2n}}{2n}, \ \frac{\chi^2_{1-\alpha/2,2n}}{2n} \right) \tag{34}$$

and in terms of decibels,

$$CI\left[10\log_{10}\left(\frac{\overline{X}}{\theta} \right) \right] = \left[10\log_{10}\left(\chi^2_{\alpha/2,2n} / 2n \right), \ 10\log_{10}\left(\chi^2_{1-\alpha/2,2n} / 2n \right) \right] \tag{35}$$

This expression can be computed using a spreadsheet program.[5] We should note that the confidence interval is not symmetric about $\overline{X}/\theta$, so expressing the confidence interval as +/- some value in dB is an approximation. A more rigorous way to express the confidence interval is to use the entire interval, or better yet, explicitly identify the lower and upper limits.

In practice, the user typically starts with a desired confidence interval and confidence level and computes the minimum number of required samples. Table 2 lists some minimum sample sizes for the three most popular confidence levels and several confidence intervals.

[5] A caution when using Microsoft Excel: Excel computes the Chi-squared distribution as $P(X>x)$ rather than the $P(X<x)$, so some manipulation is required to get the correct answer. The Excel function name is "CHIINV".

Table 2 - Number of Samples Required (Exponential distribution, CI = full interval for Xbar/θ)			
Confidence Interval (CI), dB	Confidence Level		
	90%	95%	99%
1.50 dB	91	129	222
1.75 dB	67	95	164
2.00 dB	52	73	126
2.50 dB	33	47	81
3.00 dB	23	33	56

Because the confidence interval in dB is not symmetric about the mean, we have listed the entire confidence interval in the first column of Table 1. For example, a confidence interval of 2.0 dB is approximately equal to Lee's +/- 1 dB confidence interval. Note that the values do not differ significantly from the TSB-88-A equations on page 90. TSB-88-B, page 123, uses Lee's earlier expression [3], which is incorrect. The TSB-88-B expression requires 36 samples for a 90% confidence interval of +/- 1 dB while the correct expression requires 52 samples for the same size confidence interval.

A normal approximation to the confidence interval is derived in Appendix A. For large n($n > 20$) it agrees very closely with the exact expression and with Lee's approximation from [2]. Thus, for large n, any of the three approaches results in an accurate number of samples, but (35) above is the exact solution and works for all values of n.

5.0 Mean vs. Median

When collecting field measurements of fading signals, a common concern is that the sensitivity of the receiver will affect the accuracy of the estimate of the mean. For example, if the actual mean is -117 dBm and the noise floor of the receiver is -120 dBm, the receiver will record a power level that is 50% higher than actual, resulting in an error of 1.8 dB.

Some have proposed to solve this problem by using the median rather than the mean. There are problems with using the median:

- The mean and median of the are not the same for either the Rayleigh or exponential distributions. For example, the median of the exponential distribution is 1.6 dB below the mean.
- The vendor has specified the radio's performance in Rayleigh fading with a particular *mean*, not median.
- An good unbiased and low variance estimator for the median of the Rayleigh or the exponential distribution is not known (at least not to this author). Even if a good

estimator was known, the corresponding confidence interval and therefore the minimum required number of samples may differ widely from the result derived above.

Although the median might be the preferred figure of merit when the test receiver has limited dynamic range, this is rarely a problem in practice. Typically, we are interested in service thresholds on fading channels of –105 dBm and higher, but most test receivers have noise floors of –120 dBm or lower. This difference of 15 dB creates only a very small error in the estimate of the mean (about 0.1 dB). See Appendix B for further discussion of this topic.

6.0 References

[1] W.C. Jakes, ed., Microwave Mobile Communications, IEEE Press Reissue, 1994.

[2] W.C.Y. Lee, Mobile Communications Design Fundamentals, 2nd Ed., Wiley, 1993.

[3] W.C.Y. Lee, "Estimate of Local Average Power of a Mobile Radio Signal," *IEEE Transactions on Vehicular Technology*, February, 1985, pp. 22-27.

[4] R. J. Larsen, M. L. Marx, An Introduction to Mathematical Statistics and its Applications, Prentice-Hall, 1986, pp. 281.

[5] S. Ross, A First Course in Probability, 2nd Ed., MacMillan, 1984.

[6] M. M. Peritsky, "Statistical Estimation of Mean Signal Strength in a Rayleigh Fading Environment," *IEEE Transactions on Communications*, November, 1973, pp. 1207-1213.

[7] G. C. Hess, Land Mobile Radio System Engineering, Artech House, 1993.

[8] EIA TSB-88-A, "Wireless Communications Systems — Performance in Noise and Interference-Limited Situations, Recommended Methods for Technology-Independent Modeling, Simulation and Verification," With Addendum 1, January, 2002.

[9] EIA TSB-88-B, "Wireless Communications Systems — Performance in Noise and Interference-Limited Situations, Recommended Methods for Technology-Independent Modeling, Simulation and Verification," With Addendum 1, May, 2005.

[10] D. Wong and D.C. Cox, "Estimating Local Mean Signal Power Level in a Rayleigh Fading Environment," *IEEE Transactions on Vehicular Technology*, May, 1999, pp. 956-959.

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Appendix A - Normal Approximation to the Exponential Confidence Interval

For large n, an alternate expression for the confidence interval can be derived using the central limit theorem and the normal approximation. Recall that

$$P\left(\frac{W-\mu}{\sigma/\sqrt{n}} \le a\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx \qquad \text{as } n \to \infty \tag{A.1}$$

Accordingly, we can write the 1-α confidence level as

$$CL = (1-\alpha) = P\left(-z_{\alpha/2} < \frac{W-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) \tag{A.2}$$

where $z_{\alpha/2}$ is the argument of the unit normal distribution for a confidence level of 1-α. I.e., $P(Z< z_{\alpha/2}) - P(Z< -z_{\alpha/2}) = 1-\alpha$. The most commonly used values of $z_{\alpha/2}$ are 1.65, 1.96 and 2.58 for confidence levels of 90%, 95%, and 99%, respectively.[6]

We can rewrite (A.2) in terms of μ,

$$CL = (1-\alpha) = P\left(W - \frac{z_{\alpha/2}\sigma}{\sqrt{n}} < \mu < W + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) \tag{A.3}$$

The estimator, W, is the sample mean,

$$W = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

So the confidence interval for μ is

$$CI(\mu) = \left(\overline{X} - \frac{z_{\alpha/2}\sigma}{\sqrt{n}},\ \overline{X} + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) \tag{A.4}$$

This expression is not entirely satisfactory because it includes the standard deviation, which for the exponential distribution is equal to the mean. A more useful confidence interval is one for the ratio of $\overline{X}$ to μ. Rearranging the terms of (A.3) and noting that $W = \overline{X}$, we get

$$CL = (1-\alpha) = P\left(\mu - \frac{z_{\alpha/2}\sigma}{\sqrt{n}} < \overline{X} < \mu + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right)$$

[6]Note: In Excel, NORMSINV(.05) is -1.65 and NORMSINV(.95) is +1.65. Thus, our definition of z(alpha/2) is not identical to NORMSINV(alpha/2). This version of z(alpha/2) is consistent with Larsen and Marx, but not with Excel. One can manipulate the use of the Excel function to get the correct answer. Previously, with the Chi-squared distribution, we used z(alpha/2) and z(1-alpha/2) which does match Excel, but with the inverse cdf, $P(X>x)$ rather than the cdf, $P(X<x)$.

Dividing each side by μ,

$$CL = (1-\alpha) = P\left(1 - \frac{z_{\alpha/2}\sigma}{\sqrt{n}\mu} < \frac{\overline{X}}{\mu} < 1 + \frac{z_{\alpha/2}\sigma}{\sqrt{n}\mu}\right) \tag{A.5}$$

To this point, our expression is general and applies to nearly all distributions. In our particular case, however, the mean and standard deviation of the exponential distribution are identical and the ratio of σ to μ is 1. Thus, (A.5) simplifies to

$$CL = (1-\alpha) = P\left(1 - \frac{z_{\alpha/2}}{\sqrt{n}} < \frac{\overline{X}}{\mu} < 1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right) \tag{A.6}$$

and the confidence interval is

$$CI\left(\frac{\overline{X}}{\mu}\right) = \left(1 - \frac{z_{\alpha/2}}{\sqrt{n}},\ 1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right) \tag{A.7}$$

In terms of decibels,

$$CI\left[10\log_{10}\left(\frac{\overline{X}}{\mu}\right)\right] = \left[10\log_{10}\left(1 - \frac{z_{\alpha/2}}{\sqrt{n}}\right),\ 10\log_{10}\left(1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right)\right],\ z_{\alpha/2} < \sqrt{n} \tag{A.8}$$

At a 90% confidence level, this normal approximation to the confidence interval is within 0.1 dB of the exact interval for *n*>22.

Lee's Confidence Interval. Using the same approach as above, Lee derives the following expression for the confidence interval of a Rayleigh distributed random variable in [2]:

$$CI\left(\frac{\overline{X}}{\mu}\right) = \left(1 - \frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{\frac{4-\pi}{\pi}},\ 1 + \frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{\frac{4-\pi}{\pi}}\right) \tag{A.9}$$

In terms of decibels,

$$CI\left[20\log_{10}\left(\frac{\overline{X}}{\mu}\right)\right] = \left[20\log_{10}\left(1 - \frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{\frac{4-\pi}{\pi}}\right),\ 20\log_{10}\left(1 + \frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{\frac{4-\pi}{\pi}}\right)\right] \tag{A.10}$$

Note that we are now working with amplitude ratios, not power ratios, so the factor of 20 is used instead of 10. Interestingly, this confidence interval does not differ significantly from the exact expression for the exponential distribution.

Appendix B - Measurement Error from Limited Receiver Dynamic Range

The purpose of this appendix is to quantify the error introduced by the noise floor of a test receiver when receiver measurements are used to estimate the mean of a sampled random variable with an exponential distribution. The exponential distribution is chosen because it is the distribution of the received power on a Rayleigh fading channel.

The exponential distribution has the following power density function

$$f_X(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right), \quad x \geq 0, \tag{B.1}$$

where the parameter of the distribution, θ, is also the mean.

The receiver measures the sum of the signal and the thermal noise power. It will be convenient to express the average noise power as some fraction of the mean signal power. Mathematically, we can describe the sampling process as follows:

$$g(x) = x + a\theta, \; 0 \leq x < \infty \tag{B.2}$$

where a is a constant such that $a \geq 0$. Note that we are approximating the noise power by its average value. In reality, the noise amplitude is also random with a Gaussian distribution.

The mean value of the sampled signal, $g(x)$, is given by

$$E[g(x)] = \int_0^\infty (x + a\theta) f_X(x)dx = E[X] + a\theta = \theta(a+1) \tag{B.3}$$

Thus, the error in decibels is simply $10\log_{10}(a + 1)$. This value is listed in Table B.1 for various values of a in dB. Note that the error is always positive, meaning that the receiver will record a stronger signal due to the added noise power. Table B.1 shows that as long as the service threshold of interest is more than 6 dB above the thermal noise floor of the receiver, the measurement error will be less than 1.0 dB.

Table B.1 - Measurement Error	
Thermal Noise Floor Relative to Mean Signal, dB	**Measurement Error, dB**
10	10.4
6	7.0
3	4.8
0	3.0
-1	2.5
-2	2.1
-3	1.8
-6	1.0
-10	0.4
-15	0.1
-20	0.04
-30	0.004